which finally reduces to

$$\frac{1}{a_{r+1}-a_1}\prod_{\substack{j=2\\ j\neq i}}^{r+1}\frac{1}{a_i-a_j}.$$

Thus (3) may be written

$$N=\prod_{j=1}^{r}\frac{1}{a_{r+1}-a_j}+\frac{1}{a_{r+1}-a_1}\sum_{i=2}^{r}\prod_{\substack{j=2\\ j\neq i}}^{r+1}\frac{1}{a_i-a_j}$$

$$=\frac{1}{a_{r+1}-a_1}\left[\prod_{j=2}^{r}\frac{1}{a_{r+1}-a_j}+\sum_{i=2}^{r}\prod_{\substack{j=2\\ j\neq i}}^{r+1}\frac{1}{a_i-a_j}\right]$$

$$=\frac{1}{a_{r+1}-a_1}\sum_{i=2}^{r+1}\prod_{\substack{j=2\\ j\neq i}}^{r+1}\frac{1}{a_i-a_j}.$$

Then using relation (1) in form applicable to the set $(a_i \mid i = 2, 3, \cdots, r+1)$ we see that N is zero and, by induction, relation (I) holds as stated.

This proof of a simple relation between numbers is of interest in an elementary way in algebra. While the theorem may be of no particular value in applications, its proof by induction brings in certain elements which are not often found in a proof by that method. The ordinary assumption is used twice in the proof, whereas, in most, if not all, other induction proofs it is used only once.

A simple proof of the theorem may be given if we presuppose the complex variable theory. For, if we have a set of distinct numbers $(a_1, a_2, \cdots, a_n)$, $n > 1$, the quantity

$$\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{a_i-a_j}$$

is, for every i, just the residue at the point a_i of

$$\prod_{j=1}^{n}\frac{1}{z-a_j}$$

and the sum of these residues is zero, so that the theorem is immediate.

II. RELATING TO THE FOLIUM OF DESCARTES.

By Maurice C. Baudin, Student at The University of Chicago.

The current methods of tracing the cubic

$$x^3 + y^3 - \mu xy = 0$$

are very complicated; so complicated indeed that their authors have found it too laborious to apply them thoroughly and draw a correct figure.[1] Dr. G. Teixeira in his *Obras sobre Mathematica* proposes a process by which, with a sensible amount

[1] In Dowling and Turneaure's *Analytic Geometry* the curve seems to have besides the double point at the origin, a cusp at P.

of algebraic work, we can obtain one point on the curve; and the same analysis must be repeated for every point. Other examples found in textbooks are of the same character. To remedy this I offer the following somewhat simpler and shorter method for plotting the Folium.

Let there be any right angle with its vertex at the origin of the cubic. Its sides

$$y - \lambda x = 0, \quad \lambda y + x = 0$$

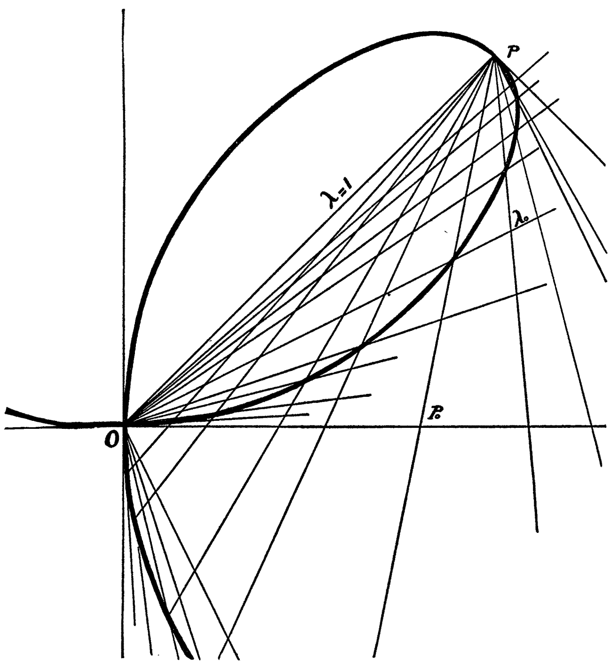
intersect the curve in two distinct points,

$$P_1\left(\frac{\mu\lambda}{1+\lambda^3}, \frac{\mu\lambda^2}{1+\lambda^3}\right), \quad P_2\left(\frac{\mu\lambda^2}{1-\lambda^3}, \frac{\mu\lambda}{\lambda^3-1}\right),$$

which determine the line

(1) $$f(x, y, \lambda) = (\lambda^2 + \lambda - 1)y - (\lambda^2 - \lambda - 1)x - \mu\lambda = 0,$$

where λ is a constant. If this right angle rotates about O, then $f(x, y, \lambda) = 0$, where λ now is the independent variable, is the equation of the family of straight lines P_1P_2 corresponding to the different values of λ. Let us determine the



envelope of these lines. Setting

(2) $$\frac{\partial f}{\partial \lambda} = 0$$

and eliminating λ from (1) and (2), we get

$$5x^2 + 5y^2 - 6xy - 2\mu x - 2\mu y + \mu^2 = 0,$$

which is the equation of a null ellipse centered at P $(\mu/2, \mu/2)$. Indeed this can be seen from equation (1), which may be written

$$y - x + \rho(2x - \mu) = 0,$$

where

$$\rho = \frac{\lambda}{\lambda^2 + \lambda - 1},$$

and is evidently satisfied for

$$x = y = \frac{\mu}{2}.$$

Therefore, $f(x, y, \lambda) = 0$ represents a pencil of lines through P. We can determine completely any one of these lines, $f(x, y, \lambda_0) = 0$, by its intercept

$$x = \frac{\mu\lambda_0}{1 + \lambda_0 - \lambda_0^2}.$$

Hence, we draw the right angle $(y - \lambda_0 x)(\lambda_0 y + x) = 0$, and plot

$$P_0 \left(\frac{\mu\lambda_0}{1 + \lambda_0 - \lambda_0^2}, 0\right),$$

and through P and P_0 we draw the line $f(x, y, \lambda_0) = 0$. The intersections of this line with $(y - \lambda_0 x)(\lambda_0 y + x) = 0$ are two points on the curve. So are the intersections of every line $f(x, y, \lambda_k) = 0$ through P_k $[\mu\lambda_k/(1 + \lambda_k - \lambda_k^2), 0]$ and P with $(y - \lambda_k x)(\lambda_k y + x) = 0$. P itself evidently is a point on the curve. Since the equation of the cubic is symmetrical in x and y the curve has the bisector $y - x = 0$ of the coördinate axes for its axis of symmetry; and we may confine ourselves to values of λ between 0 and 1.

A CORRECTION.

A correspondent has kindly pointed out that my recent Simple Proof of Hart's Theorem which appeared in the February number of the MONTHLY was wrongly named because though it was simple and had to do with Hart's theorem, it was not a proof. The correspondence of great circle to pole is, of course, one to two. I naïvely assumed that the pairs of circles corresponding to given circles would fall into two distinct groups and that one could confine one's attention to one of them. Unfortunately, in many cases, the two are hopelessly twisted together. If any other reader of the MONTHLY beside my correspondent has read the article, I hereby offer him my apology.

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